Reducibility by continuous functions and Wadge Degrees - Abstract

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Reducibility by continuous function
Descriptive set theory is in essence the study of the complexity of subsets of the Baire space ω^ω, the "logician's reals". This work is devoted to a natural mesure of the relative complexity of subsets of the Baire space, namely the *reducibility by continuous function*. Given two subsets A and B of the Baire space, A is said to be reducible to B, and we write $A \leq_W B$, if and only if A is the preimage of B for some continuous function f from the Baire space to itself. If we understand the complexity of A to mean the difficulty of determining membership in A, we observe that if A is reducible to B then A is, in a certain sense, no more complicated than B. Indeed, given $x \in \omega^\omega$, to determine if x is in A we only need to compute the image $f(x)$ of x by f, and determine whether it is in B or not. The problem of determining membership in A can therefore be reduced to that of determining membership in B, provided that the computation of $f(x)$ introduces no extra complexity. Another motivation for studying the reducibility by continuous function can be found in the equivalence relation induced by $\leq_W$. These equivalence classes, called the *Wadge degrees*, are indeed the smallest classes of subsets which are closed under continuous preimages. In this sense, Wadge degrees can be considered as a refinement of the well-known Borel and projective classes.

Wadge hierarchy
The relation $\leq$ on Wadge degrees is, merely by definition, a partial preordering called the Wadge ordre. If we restrict ourselves to a class Γ of subsets of the Baire space with appropriate closure and determinacy properties (e.g. Borel sets or projective sets under projective determinacy), it is in fact a

well-founded partial ordering on Wadge degrees. This follows from two important results: the Wadge's Lemma [5] and the Martin-Monk Theorem [4]. Both are proved using a very powerful correspondance between the reducibility by continuous function and a certain infinite two players game called the *Wadge Game*. Given two subsets A and B of the Baire space, we define the associated Wadge Games $W(A, B)$ to be the game in which players play alternatively elements of ω, building two infinite sequences, $x = (x_0, x_1, \dots) \in \Lambda^\omega$ for player I and $y = (y_0, y_1, \dots) \in \Lambda^\omega$ for player II. Player II is allowed to skip, even ω times, provided she also plays ω times, and wins if and only if $(x \in A \Leftrightarrow y \in B)$. We have then that II has a winning strategy in $W(A, B)$ if and only if $A \leq_W B$. Following directly from this characterization, the Wadge's Lemma states that if we restrict ourselves to a class Γ of subsets of the Baire space with appropriate closure and determinacy properties, then for all A and B in Γ, $A \leq_W B$ or $B \leq_W \omega^\omega \backslash A$. The Wadge Lemma implies that for the partial order $\leq$, any antichain has a size at most 2: a given Wadge degree $[A]_W$ is comparable to any other Wadge degree, except $[\Lambda^\omega \backslash A]_W$ if they are not equal. The Martin-Monk Theorem shows that the relation $\leq_W$ is well-founded on Γ, so that $\leq_W$ induces a hierarchy on Γ.

Results for the real line

In contrast, for topological spaces that are not zero dimensional, the structure of the Wadge order may be more chaotical [3]. The situation is completely different from the case of the Baire space and it is not possible to get the same kind of game characterization. For example, Wadge's Lemma fails and the Wadge order for the real line is ill-founded. In fact, Ikegami proved in [1] that there is an embedding i from $(\mathscr{P}(\mathbb{N}), \subseteq_{fin})$ to $(\mathscr{P}(\mathbb{R}), \leq_W)$ such that the range of i consists of subsets of real numbers which are the difference of open sets, where $a \subseteq_{fin} b$ if $a \backslash b$ is finite for subsets a, b of $\mathbb{N}$ and $\mathbb{N} = \omega \backslash \{0\}$. It is easy to construct a descending sequence of subsets of $\mathbb{N}$ with length ω with respect to $\subseteq_{fin}$, hence the Wadge order on $\mathbb{R}$ is ill-founded. Thanks to this embedding, one can also find two sets of reals which are the difference of two open sets such that neither $A \leq_W B$ nor $B \leq_W \mathbb{R} \backslash A$ holds.

References

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