Decomposing Baire class one functions

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There is two ways to define a hierarchy of functions. On the first hand, the Baire hierarchy of functions is defined inductively from the continuous functions using the taking of pointwise limit of a sequence of functions. On the other hand, the Borel hierarchy of functions is based on the complexity of inverse images of open sets in the Borel hierarchy of sets. Lebesgue has showed an exact correspondence between these two hierarchies. Here we focus on the first level of these hierarchies, the set of functions of Baire class one, and more specifically in polish spaces. Using game theoretical methods we will refine the hierarchy on this level and see on an example that the correspondence showed by Lebesgue does not hold any more.

A space is *polish* whenever it is separable and completely metrizable. In other words, when it admits both a metric such that every cauchy sequence converges and a countable dense subset. Given A and B two polish spaces, a function $f : A \to B$ is *of Baire class one* if f is the pointwise limit of a sequence of continuous functions, or equivalently if the inverse image of an open, or Σ^0_1 subset of B is a countable union of closed sets, or Σ^0_2, in A.

For a specific subset of the first Baire class there is a result binding complexity in terms of inverse images and partitions in continuous functions. Indeed the Jayne-Rogers theorem states that for A and B polish and $f : A \to B$ a function, the inverse image of a Σ^0_2 is Σ^0_2 if and only if there is a countable partition $(A_i)_{i\in\mathbb{N}}$ of A in closed sets such that for all $i \in \mathbb{N}$ the restriction of f to A_i is continuous. Following this result Andretta proved that for A and B totally disconnected polish spaces, Baire class one functions can be represented as strategies in infinite games.

The Baire space is the set of all infinite sequences of integers along with the product topology, it is denoted ω^ω or $\mathcal{N}$. It is canonical amongst totally disconnected polish spaces, because any such space is homeomorphic a closed subset of the Baire space. Given a function $f : \mathcal{N} \to \mathcal{N}$, we consider the following two-payer infinite game with perfect information. Player I and II play at each turn an integer, thus building in ω turns two elements of $\mathcal{N}$, x and y respectively. Player II wins if and only if $f(x) = y$. If player II is allowed to skip his turn then he has a winning strategy if and only if the function f is continuous. If in addition he is allowed to erase what he has done and restart from scratch then II has a winning strategy if and only if the inverse image of a Σ^0_2 is Σ^0_2.

We call this last game the *backtrack* game. Of course, in a play of this game player II can erase only finitely many times if he wants to produce an infinite sequence. So given a function $f : \mathcal{N} \to \mathcal{N}$ and a winning strategy for player II in the backtrack game we can bound the number of times II erases in all plays by a countable ordinal. Using this bound we define the *discontinuity* of a function f. This give us a procedure to find a partition $(A_i)_{i\in\mathbb{N}}$ of A in closed sets such that $f|_{A_i}$ is continuous, inducing a hierarchy of Baire class one functions. The corresponding refinement of the Borel hierarchy of sets is the Wadge hierarchy of Σ^0_2-sets. We observe that although there is a close relation between the Wadge degree of a set and the discontinuity of its characteristic function, there is no exact correspondence between the discontinuity of a function and the Wadge degree of inverse images of open sets. This construction also leads to a game-theoretical proof of the Jayne-Rogers theorem.

References

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