Constructing the Lindenbaum algebra for a logic step-by-step using duality

Sam van Gool*

November 15, 2010

This abstract is on joint work with Dion Coumans and Mai Gehrke.

In the study of a propositional logic $\mathcal{L}$, the following construction is often important: take the set of all propositional formulas in the language of $\mathcal{L}$, and partition this set into classes of $\mathcal{L}$-equivalent formulas. The set of $\mathcal{L}$-equivalence classes has a natural algebraic structure, which is called the *Lindenbaum algebra* for the logic $\mathcal{L}$. Algebraic methods are useful to study issues such as term complexity, decidability of logical equivalence, and normal forms for a logic, i.e., problems in which one considers formulas whose variables are drawn from a finite set.

The oldest instance of the use of algebraic methods in logic goes back to George Boole: the Lindenbaum algebra for *classical propositional logic* (CPL) on n variables $\{p_1, \ldots, p_n\}$ is a *Boolean algebra*, and it can be shown to be (isomorphic to) $\mathcal{P}(\mathcal{P}(\{p_1, \ldots, p_n\}))$. The logical impact of this result is the disjunctive normal form theorem for CPL.

However, for logics other than CPL, the situation is often much more complicated, and a simple description of the Lindenbaum algebra is usually not available. For example, the Lindenbaum algebra for *intuitionistic propositional logic* (IPL) on only two variables is already infinite and non-trivial to describe.

Modal logics form another rich class of examples of logics whose Lindenbaum algebras are often infinite and complicated. These logics are based on CPL, enriched with a unary connective '$\Box$', which is meant to formalize a notion of 'necessity'. Different axioms for $\Box$ yield different modal logics. One may try to gain a better understanding of a particular modal logic through its Lindenbaum algebra. As a representative example, we will mainly

*Institute for Mathematics, Astrophysics and Particle Physics, Radboud Universiteit Nijmegen, P.O. Box 9010, 6500 GL Nijmegen, The Netherlands. E-mail: s.vangool@math.ru.nl

concentrate on the Lindenbaum algebra for a modal logic called $\mathsf{S4}$, but we will also indicate how the methods described may apply to a larger class of logics.

At first sight, the formulas of modal logic look similar to those for CPL, but the addition of the operation $\Box$ will usually complicate matters quite a bit. The following definition measures 'how modal' a particular formula φ in the language of modal logic is: we say φ has *rank* i if all propositional variables in φ are under the scope of exactly i occurrences of $\Box$.

We can use this definition to understand the Lindenbaum algebra on n variables for a modal logic $\mathcal{L}$ in a layered manner: the classes of formulas of that are Boolean combinations of formulas of rank at most i in the Lindenbaum algebra form a Boolean algebra $\mathbb{B}_i$, and the Lindenbaum algebra is then the union $\mathbb{B}_0 \cup \mathbb{B}_1 \cup \mathbb{B}_2 \cup \cdots$. Each of the algebras $\mathbb{B}_i$ will be finite, so this method provides a way of approximating the infinite Lindenbaum algebra $\mathbb{B}$ by its finite pieces. In technical terms, the Lindenbaum algebra can be shown to be the *direct limit* of a sequence of finite algebras (see, e.g., [2]).

In certain well-behaved examples, those finite pieces $\mathbb{B}_0, \mathbb{B}_1, \ldots$ of the Lindenbaum algebra can be described by an *inductive definition* of the following kind: start from a simple $\mathbb{B}_0$ (usually the Lindenbaum algebra for CPL), and then define $\mathbb{B}_{n+1}$ from $\mathbb{B}_n$ in a uniform way. For instance, if all the axioms for the logic under consideration are of rank 1, then it can be shown that such an easy description of the Lindenbaum algebra always exists. Such a description, in turn, will then also yield a normal form theorem for the logic $\mathcal{L}$ similar to the one for CPL.

In recent work, Ghilardi [3], [4] and N. Bezhanishvili and Gehrke [1] have widened the scope of this 'constructive step-by-step approach' to logics that are not axiomatizable by rank 1 axioms, namely IPL and $\mathsf{S4}$. A key feature of these authors' approaches is the use of *duality* to obtain a concrete description of the finite algebras in the approximating sequence.

We will discuss the representative example of the step-by-step construction of the Lindenbaum algebra for $\mathsf{S4}$ via duality, based on [4] and our original work. We will also indicate how our understanding of the $\mathsf{S4}$ case helps to shed light on the answer to an open conjecture, namely that the step-by-step method can be generalized to all logics whose axiomatization is of *mixed rank 0 and 1.*

References

[1] Nick Bezhanishvili and Mai Gehrke, *Finitely generated free Heyting algebras via Birkhoff duality and coalgebra*, to appear in Logical Methods in Computer Science.

[2] Nick Bezhanishvili and Alexander Kurz, *Free modal algebras: a coalgebraic perspective*, in: Algebra and Coalgebra in Computer Science, Lecture Notes in Computer Science, vol. 4624, Springer, 2007, pp. 143–157.

[3] Silvio Ghilardi, *Free Heyting algebras as bi-Heyting algebras*, Math. Rep. Acad. Sci. Canada **XVI** (1992), no. 6, 240–244.

[4] ——, *Continuity, Freeness, and Filtrations*, Rapporto Interno 331-10, Dipartimento di Scienze dell'Informazione, Università degli Studi di Milano (2010).