Axioms for Non-Archimedean Probability (NAP)

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In this contribution, we focus on probabilistic problems with a denumerably or non-denumerably infinite number of possible outcomes. Kolmogorov (1933) provided an axiomatic basis for probability theory, presented as a part of measure theory, which is a branch of standard analysis or calculus. Since standard standard analysis does not allow for non-Archimedean quantities (*i.e.* infinitesimals), we may call Kolmogorov's approach 'Archimedean probability theory'. We show that allowing non-Archimedean probability values may have considerate epistemological advantages in the infinite case.

Kolmogorov deals with infinite outcome spaces primarily by the Axiom of Continuity. In combination with finite additivity, this axiom leads to σ- or countable additivity in cases where the event space is a σ-algebra. However, the Axiom of Continuity and the property of countable additivity (CA) give rise to a number of epistemological issues, such as:

(1) An event with probability 1 is not necessarily certain to occur.

(2) An event with probability 0 is not necessarily impossible to occur.

(3) It turns out to be impossible to model certain simple problems, such as a fair lottery on the natural numbers.

Issue (3) can be resolved by dropping the normalization requirement (Rényi, 1955) or dropping CA (de Finetti, 1974). It has also been suggested (e.g. by Skyrms, 1980, Lewis, 1986, Elga, 2004) to allow infinitesimals in the domain of the probability function. This suggestion has been applied explicitly to the case of a fair lottery on the natural numbers (Wenmackers and Horsten, 2010). For this case, also issues (1) and (2) are resolved.

In this study, we generalize the solution given by Wenmackers and Horsten (2010) to probabilistic problems not only on other countably infinite sample spaces (*e.g.* a fair lottery on $\mathbb{Q}$), but on sample spaces of larger cardinalities as well (*e.g.* a fair lottery on $\mathbb{R}$).

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Since the range of the probability function may contain infinitesimals in our approach, we call it Non-Archimedean Probability (NAP). These are the Axioms of NAP, where Ω is a set called the sample space:

NAP0 The probability function P has as its domain the full powerset of Ω (event space = $\mathcal{P}(\Omega)$) and as its range the unit interval of a suitable, ordered field F (range = $[0,1]_F$).

NAP1 $P(A) = 0 \Leftrightarrow A = \emptyset$

NAP2 $P(A) = 1 \Leftrightarrow A = \Omega$

NAP3 For all events $A, B \in \mathcal{P}(\Omega)$ such that $A \cap B = \emptyset$:

$$P(A \cup B) = P(A) + P(B).$$

NAP4 There exists a directed set $\langle \Lambda, \subseteq \rangle$[1] with $\Lambda \subseteq \mathcal{P}_{fin}(\Omega)$[2] such that:

$$\forall E \in P_{fin}(\Omega), \exists \lambda \in \Lambda : E \subset \lambda.$$ [3]

Regarding the range of P, the corresponding axiom of Kolmogorov requires it to be a σ-algebra over Ω, which may be strictly smaller than the full power set; NAP0 ensures that every set in $\mathcal{P}(\Omega)$ represents an event. Whereas Kolmogorov fixes the range of all probability functions (as $[0,1]_{\mathbb{R}}$), in our case, the range depends on the problem (in particular, the sample space Ω). Moreover,

NAP1 is intended to formalize the philosophical notion called 'Regularity' (Elga, 2004). The pair NAP1 & NAP2 makes it possible to interpret probability 0 and 1 events safely as impossible and necessary, respectively. (In Kolmogorov's system, only the $\Leftarrow$ direction holds.)

The axiom NAP3 is exactly the same as Kolmogorov's addition-rule for events (finite additivity).

NAP4 takes the place of the Continuity Axiom and thus represents the most drastic difference as compared to Kolmogorov's approach. Just like the Continuity Axiom implies the use of classical limits, NAP4 implies the use of a generalized limit (direct limit). In the case of a denumerably infinite sample space, this limit turns out to be the α-limit, as defined by Benci and Di Nasso (2003) in the context of Alpha-Theory, an axiomatic approach to non-standard analysis (NSA). Non-standard models can be obtained in terms of maximal ideals or free ultrafilters; the use of the directed set $\langle \Lambda, \subseteq \rangle$ has the advantage that unlike the other two options, Λ can be

[1] $\langle \Lambda, \subseteq \rangle$ is a directed set if and only if Λ is a non-empty set and $\subseteq$ is a preorder such that every pair of elements of Λ has an upper bound: $(\forall A, B \in \Lambda)\exists C \in \Lambda : A \cup B \subseteq C$.

[2] $\mathcal{P}_{fin}(\Omega)$ denotes the family of *finite* subsets of Ω.

[3] This condition implies that $\bigcup_{\lambda \in \Lambda} \lambda = \Omega$.

stated explictely and can be chosen such as to model 'the physics' of the problem.

If time permits, we will illustrate the axioms by applying them to examples of infinite lotteries.

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