Bivalent Logics

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Logic is traditionally defined according to underlying principles. Among them, three seem particularly important. The principle of bivalence says that there are exactly two truth values, usually called True and False. The principle of excluded middle states that a sentence has at least one truth value. The principle of non-contradiction states that a sentence has at most one truth value. A logic that satisfies the conjunction of these three principles is called classical. By contrast, a logic is called non-classical if it does not support at least one of them.

In relation to these principles, three bivalent logics differ from classical logic insofar as they ignore the principle of excluded middle and/or the principle of non-contradiction : paraconsistent logic satisfies the principle of excluded middle, paracomplete logic satisfies the principle of non-contradiction and positive logic ignores both of these principles. In addition to classical logic, three bivalent (non-classical) logics can therefore be distinguished.

My purpose is to provide a unified framework for studying the semantic and syntactic relationships between these four bivalent logics. More specifically, my aim is to characterize the notion of logical consequence within each of these logics. To do this, I propose a new definition of the notions of model and sequent which makes explicit these principles.

The syntactic approach I have chosen is sequent calculus. For each of the logics mentioned above, I give a notion of validity and propose an associated sequent calculus. A sequent is called valid, glut-valid, gap-valid and classic-valid if it is semantically correct in positive logic, paraconsistent logic, paracomplete logic and classical logic, respectively. Similarly, a sequent is called derivable, glut-derivable, gap-derivable and classic-derivable if it is syntactically correct in positive logic, paraconsistent logic, paracomplete logic and classical logic, respectively.

Here are some of the results I reached :

1. The class of derivable sequents is *strictly* included in that of sequents both glut-derivable and gap-derivable. In other words, if a sequent is derivable, then it is glut-derivable and gap-derivable. The converse does not hold.
2. The class of glut-derivable and/or gap-derivable sequents is *strictly* included in the class of classic-derivable sequents. It means that if a sequent is glut-derivable and/or gap-derivable, then it is classic-derivable. The converse does not hold.
3. Starting with an adequate definition of the dual of a sequent, I show that a sequent is derivable if and only if its dual is derivable. The same result can be obtained in the case of the classic-derivability. On the other hand, I show that a sequent is glut-derivable if and only if its dual is gap-derivable and that a sequent is gap-derivable if and only if its dual is glut-derivable.
4. Four different forms of the original cut elimination rule are distinguishable in the sequent calculi mentioned above. Only two hold for the positive sequent calculus while all of them hold for the classical sequent calculus. As for the paraconsistent and paracomplete sequent calculi, they only admit one form of the cut elimination rule in addition to two that hold for the positive sequent calculus.
5. I provide a proof for the soundness and the completeness theorems that assert the extensional equivalence of the semantic and syntactic definitions of the sequent correctness. In other words, I show that the class of valid, glut-valid, gap-valid and classic-valid sequents is identical to that of derivable, glut-derivable, gap-derivable and classic-derivable sequents, respectively.