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Title: Regularity properties and the projective hierarchy.

Abstract: In the study of the real number continuum, *regularity properties* of sets of real numbers play a central role, having many applications in various areas of mathematics. The concept of a subset of the continuum being *Lebesgue-measurable*, for instance, is motivated by the attempt to formalize "size" and "volume" of objects in space. The *Baire property* is another concept motivated by topological issues; the *Ramsey property* is based on the extension of the finite Ramsey theorem to infinite dimensions. These are just three examples of a wide array of regularity properties.

Using the well-ordering of the continuum (which follows from the axiom of choice) one can construct sets that are irregular, i.e., which do not satisfy any regularity property. However, one cannot give explicit examples of such sets. What exactly does that mean?

In descriptive set theory, sets of real numbers are classified in a complexity hierarchy, the simplest of which are the Borel sets, the next level being the analytic or $\boldsymbol{\Sigma}^1_1$ sets, followed by the $\boldsymbol{\Sigma}^1_2$, $\boldsymbol{\Pi}^1_2$ sets, etc. Roughly speaking, a set definable in second-order Peano Arithmetic is classified in the complexity class which corresponds to the logical complexity of the formula defining it. In this hierarchy, the analytic sets can be shown to satisfy all regularity properties, and so the first levels on which we may find irregularities are the $\boldsymbol{\Sigma}^1_2$-, $\boldsymbol{\Pi}^1_2$- and $\boldsymbol{\Delta}^1_2$-levels. Typically, the statement "all $\boldsymbol{\Delta}^1_2/\boldsymbol{\Sigma}^1_2$ sets are regular" is independent of ZFC, the standard axiomatisation of set theory.

Specifically, the regularity of $\boldsymbol{\Delta}^1_2$ sets fails in Gödel's constructible universe $\mathbb{L}$. On the other hand, it typically holds if we assume sufficient "transcendence" over $\mathbb{L}$, that is, if we assume that the actual set-theoretic universe is far from being $\mathbb{L}$. In fact, one can be more specific and prove that the assertion that all sets on the $\boldsymbol{\Sigma}^1_2/\boldsymbol{\Delta}^1_2$-level satisfy a certain regularity property, is equivalent to the statement that the actual universe is larger than $\mathbb{L}$ *in a certain way.* You could say that regularity on the second level of the projective hierarchy measures the amount of transcendence over $\mathbb{L}$. This characterisation is very useful because transcendence statements can, to some degree, be controlled by the method of *iterated forcing.* Starting with the model $\mathbb{L}$ one repeatedly adjoins new real numbers in such a way that, in the final model, certain transcendence statements will hold whereas others will fail. It follows that in these model, certain regularity properties will hold on the $\boldsymbol{\Delta}^1_2/\boldsymbol{\Sigma}^1_2$-level whereas others shall fail.

In this talk I will present the general theory of regularity properties for sets on the second level of the projective hierarchy, mentioning some general results and paying particualr attention to the problems involved in my own work.